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Integrable potentials on spaces with curvature from quantum groups

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Abstract

A family of classical integrable systems defined on a deformation of the two-dimensional sphere, hyperbolic and (anti-)de Sitter spaces is constructed through Hamiltonians defined on the non-standard quantum deformation of a $sl(2)$ Poisson coalgebra. All these spaces have a non-constant curvature that depends on the deformation parameter z. As particular cases, the analogues of the harmonic oscillator and Kepler–Coulomb potentials on such spaces are proposed. Another deformed Hamiltonian is also shown to provide superintegrable systems on the usual sphere, hyperbolic and (anti-)de Sitter spaces with a constant curvature that exactly coincides with z. According to each specific space, the resulting potential is interpreted as the superposition of a central harmonic oscillator with either two more oscillators or centrifugal barriers. The non-deformed limit $z \to 0$ of all these Hamiltonians can then be regarded as the zero-curvature limit (contraction) which leads to the corresponding (super)integrable systems on the flat Euclidean and Minkowskian spaces.

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1. Introduction

One of the possible applications of quantum deformations of groups and algebras [1–5] is the construction of classical and quantum integrable systems with an arbitrary number of degrees of freedom which was presented in [6]. In this context, Poisson coalgebras (Poisson algebras endowed with a compatible coproduct structure) have been shown to generate in a systematic way certain (super)integrable classical Hamiltonian systems. In this construction, once a symplectic realization of the algebra is given, the generators of the Poisson coalgebra play the role of dynamical symmetries of the Hamiltonian, while the coproduct is used to 'propagate' the integrability to arbitrary dimension. From this coalgebra approach, several well-known classical (super)integrable systems have been recovered and some integrable deformations for them, as well as new integrable systems, have also been obtained [6–11].

Recently, this integrability-preserving deformation procedure has been used to introduce both superintegrable and integrable free motions on two-dimensional (2D) spaces with curvature, either constant or variable, respectively [12]. Therefore one could expect that potential terms can also be considered, in such a way that the coalgebra approach should provide (super)integrable potentials on curved spaces. The aim of the present paper is to prove this assertion through the construction of some relevant Hamiltonians.

In order to make these ideas more explicit, let us consider the non-standard quantum deformation of $sl(2)$ [13] written as a Poisson coalgebra $(sl_z(2), \Delta_z)$ with (deformed) Poisson brackets, coproduct and Casimir given by

$$\{J_3, J_+\} = 2J_+ \cosh zJ_- \qquad \{J_3, J_-\} = -2\frac{\sinh zJ_-}{z} \qquad \{J_-, J_+\} = 4J_3 \tag{1.1}$$

$$\Delta_z(J_-) = J_- \otimes 1 + 1 \otimes J_- \qquad \Delta_z(J_i) = J_i \otimes \mathrm{e}^{zJ_-} + \mathrm{e}^{-zJ_-} \otimes J_i \qquad i = +, 3 \tag{1.2}$$

$$\mathcal{C}_z = \frac{\sinh zJ_-}{z} J_+ - J_3^2 \tag{1.3}$$

where z is a real deformation parameter. A two-particle symplectic realization of (1.1) in terms of two canonical pairs of coordinates (q_1, q_2) and momenta (p_1, p_2) with respect to the usual Poisson bracket

$$\{f, g\} = \sum_{i=1}^{2} \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i} \frac{\partial f}{\partial p_i} \right), \tag{1.4}$$

and that depends on two real parameters b_1, b_2, reads [7, 11]

$$\begin{aligned} J_- &= q_1^2 + q_2^2 \qquad J_3 = \frac{\sinh zq_1^2}{zq_1^2} q_1 p_1 \,\mathrm{e}^{zq_2^2} + \frac{\sinh zq_2^2}{zq_2^2} q_2 p_2 \,\mathrm{e}^{-zq_1^2} \\ J_+ &= \left(\frac{\sinh zq_1^2}{zq_1^2} p_1^2 + \frac{zb_1}{\sinh zq_1^2} \right) \mathrm{e}^{zq_2^2} + \left(\frac{\sinh zq_2^2}{zq_2^2} p_2^2 + \frac{zb_2}{\sinh zq_2^2} \right) \mathrm{e}^{-zq_1^2}. \end{aligned} \tag{1.5}$$

By substituting (1.5) into (1.3) we obtain the two-particle Casimir

$$\begin{aligned} \mathcal{C}_z = {} & \frac{\sinh zq_1^2}{zq_1^2} \frac{\sinh zq_2^2}{zq_2^2} (q_1 p_2 - q_2 p_1)^2 \,\mathrm{e}^{-zq_1^2} \mathrm{e}^{zq_2^2} + \left(b_1 \,\mathrm{e}^{2zq_2^2} + b_2 \,\mathrm{e}^{-2zq_1^2} \right) \\ & + \left(b_1 \frac{\sinh zq_2^2}{\sinh zq_1^2} + b_2 \frac{\sinh zq_1^2}{\sinh zq_2^2} \right) \mathrm{e}^{-zq_1^2} \,\mathrm{e}^{zq_2^2} \end{aligned} \tag{1.6}$$

which Poisson-commutes with the generators (1.5). The limit $z \to 0$ of such deformed generators leads to the well-known symplectic realization of $sl(2)$:

$$J_- = q_1^2 + q_2^2 \qquad J_3 = q_1 p_1 + q_2 p_2 \qquad J_+ = p_1^2 + \frac{b_1}{q_1^2} + p_2^2 + \frac{b_2}{q_2^2}. \tag{1.7}$$

The coalgebra approach [6] ensures that *any* smooth function $\mathcal{H}_z = \mathcal{H}_z(J_-, J_+, J_3)$ defined on (1.5) provides an integrable Hamiltonian, for which $\mathcal{C}_z$ is the constant of the motion. In this paper we shall study some choices for $\mathcal{H}_z$ that lead to Hamiltonians which are quadratic in the momenta and belong to the family

$$\mathcal{H}_z = \tfrac{1}{2} J_+ f(zJ_-) + \mathcal{U}(zJ_-) \tag{1.8}$$

where f and $\mathcal{U}$ are arbitrary smooth functions such that the $\lim_{z\to 0}\mathcal{U}(zJ_-)$ is well defined and $\lim_{z\to 0} f(zJ_-) = 1$.

Therefore integrable deformations of the free motion of a particle on the 2D Euclidean space are obtained from (1.8) by setting $b_1 = b_2 = 0$ and $\mathcal{U} = 0$. Two main representative cases appear [12]:

- The simplest *integrable* Hamiltonian
$$\mathcal{H}_z^{\mathrm{I}} = \tfrac{1}{2} J_+ \tag{1.9}$$
which defines the geodesic motion on a 2D Riemannian space with metric
$$\mathrm{d}s^2 = \frac{2zq_1^2}{\sinh zq_1^2}\,\mathrm{e}^{-zq_2^2}\mathrm{d}q_1^2 + \frac{2zq_2^2}{\sinh zq_2^2}\,\mathrm{e}^{zq_1^2}\,\mathrm{d}q_2^2 \tag{1.10}$$
and whose non-constant Gaussian curvature K is given by
$$K(q_1, q_2) = -z\sinh\left(z\left(q_1^2 + q_2^2\right)\right). \tag{1.11}$$
- The *superintegrable* Hamiltonian
$$\mathcal{H}_z^{\mathrm{S}} = \tfrac{1}{2} J_+\,\mathrm{e}^{zJ_-} \tag{1.12}$$
that leads to a Riemannian metric of constant curvature which coincides with the deformation parameter, $K = z$, namely
$$\mathrm{d}s^2 = \frac{2zq_1^2}{\sinh zq_1^2}\,\mathrm{e}^{-zq_1^2}\,\mathrm{e}^{-2zq_2^2}\,\mathrm{d}q_1^2 + \frac{2zq_2^2}{\sinh zq_2^2}\,\mathrm{e}^{-zq_2^2}\,\mathrm{d}q_2^2. \tag{1.13}$$

Consequently, the 'classical' limit $z \to 0$ corresponds to a zero-curvature limit. Section 2 is devoted to the explicit solution of the geodesic flows on all these spaces, that complete the preliminary description given in [12] and include deformations of the 2D sphere and hyperbolic spaces as well as of the (1 + 1)D (anti-)de Sitter spacetimes.

The introduction of integrable potentials with coalgebra symmetry is then analysed by making use of the function $\mathcal{U}$ and taking both parameters b_i arbitrary. In fact, any $\mathcal{U}$ such that $\lim_{z\to 0}\mathcal{U} = \beta_0 J_-$ can be interpreted as a deformation of the well-known 2D Smorodinsky–Winternitz (SW) system [14–17] formed by an isotropic harmonic oscillator with angular frequency $\omega = \sqrt{\beta_0}$ plus two 'centrifugal terms' governed by b_1, b_2:
$$\mathcal{H}^{\mathrm{SW}} = \frac{1}{2}\left(p_1^2 + p_2^2\right) + \frac{b_1}{2q_1^2} + \frac{b_2}{2q_2^2} + \beta_0\left(q_1^2 + q_2^2\right). \tag{1.14}$$
On the other hand, analogues of the Kepler–Coulomb (KC) potential can be obtained by considering any $\mathcal{U}$ such that $\lim_{z\to 0}\mathcal{U} = -\gamma/\sqrt{J_-}$ (γ is another real constant):
$$\mathcal{H}^{\mathrm{KC}} = \frac{1}{2}\left(p_1^2 + p_2^2\right) + \frac{b_1}{2q_1^2} + \frac{b_2}{2q_2^2} - \frac{\gamma}{\sqrt{q_1^2 + q_2^2}}. \tag{1.15}$$

In section 3 we shall propose the following *integrable* SW and KC systems, $\mathcal{H}_z^{\mathrm{ISW}}$ and $\mathcal{H}_z^{\mathrm{IKC}}$, on the spaces of non-constant curvature previously defined through $\mathcal{H}_z^{\mathrm{I}}$ (1.9):
$$\mathcal{H}_z^{\mathrm{ISW}} = \frac{1}{2} J_+ + \beta_0 \frac{\sinh zJ_-}{z} \tag{1.16}$$
$$\mathcal{H}_z^{\mathrm{IKC}} = \frac{1}{2} J_+ - \gamma\sqrt{\frac{2z}{\mathrm{e}^{2zJ_-} - 1}}\,\mathrm{e}^{2zJ_-}. \tag{1.17}$$

The SW potential on the spaces of constant curvature with free motion given by $\mathcal{H}_z^{\mathrm{S}}$ (1.12) will be introduced by means of the following choice for the Hamiltonian:

$$\mathcal{H}_z^{\mathrm{SSW}} = \frac{1}{2} J_+ \,\mathrm{e}^{zJ_-} + \beta_0 \frac{\sinh zJ_-}{z}\,\mathrm{e}^{zJ_-} \equiv \mathcal{H}_z^{\mathrm{ISW}}\,\mathrm{e}^{zJ_-}. \tag{1.18}$$

We already know [7, 11] that this gives rise, under (1.5), to a Stäckel-type system [18] and so determines a *superintegrable* deformation of (1.14) since, besides (1.6), there exists an additional constant of the motion given by

$$\mathcal{I}_z = \frac{\sinh zq_1^2}{2zq_1^2}\,\mathrm{e}^{zq_1^2} p_1^2 + \frac{zb_1}{2\sinh zq_1^2}\,\mathrm{e}^{zq_1^2} + \frac{\beta_0}{2z}\,\mathrm{e}^{2zq_1^2}. \tag{1.19}$$

Note that this extra integral is *not* obtained from the coalgebra symmetry of the Hamiltonian[3]. Section 4 is fully devoted to the study of $\mathcal{H}_z^{\mathrm{SSW}}$, which is shown to provide a superintegrable system containing a (curved) harmonic oscillator together with two more potential terms (either oscillators or centrifugal barriers) on the usual sphere, hyperbolic and (anti-)de Sitter spaces. The explicit potentials are analysed in detail for each particular space. We stress that we recover known results on the Riemannian spaces [19–23] but also we obtain new ones on the relativistic spacetimes. Finally, some remarks and comments close the paper.

2. Deformed geodesic motion

2.1. Integrable geodesic motion on spaces of non-constant curvature

To start with, let us consider the metric (1.10) defined by the free Hamiltonian (1.9). Let us call $z = \lambda_1^2$ and introduce another parameter $\lambda_2 \neq 0$, such that λ_1 and λ_2 can be either real or pure imaginary numbers. Then under the change of coordinates $(q_1, q_2) \to (\rho, \theta)$ defined by [12]

$$\cosh(\lambda_1\rho) = \exp\left\{z\left(q_1^2 + q_2^2\right)\right\} \equiv \mathrm{e}^{zJ_-} \qquad \sin^2(\lambda_2\theta) = \frac{\exp\left\{2zq_1^2\right\} - 1}{\exp\left\{2z\left(q_1^2+q_2^2\right)\right\} - 1} \tag{2.1}$$

the metric (1.10) and curvature (1.11) are written as

$$\mathrm{d}s^2 = \frac{1}{\cosh(\lambda_1\rho)}\left(\mathrm{d}\rho^2 + \lambda_2^2 \frac{\sinh^2(\lambda_1\rho)}{\lambda_1^2}\,\mathrm{d}\theta^2\right) \tag{2.2}$$

$$K(\rho) = -\frac{1}{2}\lambda_1^2 \frac{\sinh^2(\lambda_1\rho)}{\cosh(\lambda_1\rho)}. \tag{2.3}$$

The product $\cosh(\lambda_1\rho)\,\mathrm{d}s^2$ coincides with the metric of the 2D Cayley–Klein (CK) spaces [24–26], all of them with constant curvature $\kappa_1 \equiv -z$, provided that (ρ, θ) are identified with geodesic polar coordinates. Hence λ_1, λ_2 play the role of (graded) contraction parameters, determining the curvature and the signature of the metric, respectively.

Consequently, the metric (2.2) can be interpreted as a deformation of the CK metric through the factor $1/\cosh(\lambda_1\rho) = \mathrm{e}^{-zJ_-}$, which is responsible for the transition from the constant curvature to the non-constant one, or alternatively as a deformation of the *flat* Euclidean (λ_2 real) or Minkowskian (λ_2 imaginary) spaces, which are recovered under the

[3] In order to construct the superintegrable KC system on spaces of constant curvature, at least one of the b_1, b_2 parameters must vanish and both the specific additional 'Laplace–Runge–Lenz' integral and the appropriate $\mathcal{U}$ compatible with it should be previously obtained.

Table 1. Metric and sectional curvature of the four spaces of non-constant curvature for $\lambda_1, \lambda_2 \in \{1, \mathrm{i}\}$ with deformed $sl(2)$-coalgebra symmetry. The contraction $z = \lambda_1^2 = 0$ leads to the flat Euclidean and Minkowskian spaces.

2D deformed Riemannian spaces	(1 + 1)D deformed relativistic spacetimes
• Deformed sphere $\mathbf{S}_z^2$ $z = -1$; $(\lambda_1, \lambda_2) = (\mathrm{i}, 1)$ $\mathrm{d}s^2 = \frac{1}{\cos\rho}(\mathrm{d}\rho^2 + \sin^2\rho\,\mathrm{d}\theta^2)$ $K = -\frac{\sin^2\rho}{2\cos\rho}$	• Deformed anti-de Sitter spacetime $\mathbf{AdS}_z^{1+1}$ $z = -1$; $(\lambda_1, \lambda_2) = (\mathrm{i}, \mathrm{i})$ $\mathrm{d}s^2 = \frac{1}{\cos\rho}(\mathrm{d}\rho^2 - \sin^2\rho\,\mathrm{d}\theta^2)$ $K = -\frac{\sin^2\rho}{2\cos\rho}$
• Euclidean space $\mathbf{E}^2$ $z = 0$; $(\lambda_1, \lambda_2) = (0, 1)$ $\mathrm{d}s^2 = \mathrm{d}\rho^2 + \rho^2\,\mathrm{d}\theta^2$ $K = 0$	• Minkowskian spacetime $\mathbf{M}^{1+1}$ $z = 0$; $(\lambda_1, \lambda_2) = (0, \mathrm{i})$ $\mathrm{d}s^2 = \mathrm{d}\rho^2 - \rho^2\,\mathrm{d}\theta^2$ $K = 0$
• Deformed hyperbolic space $\mathbf{H}_z^2$ $z = 1$; $(\lambda_1, \lambda_2) = (1, 1)$ $\mathrm{d}s^2 = \frac{1}{\cosh\rho}(\mathrm{d}\rho^2 + \sinh^2\rho\,\mathrm{d}\theta^2)$ $K = -\frac{\sinh^2\rho}{2\cosh\rho}$	• Deformed de Sitter spacetime $\mathbf{dS}_z^{1+1}$ $z = 1$; $(\lambda_1, \lambda_2) = (1, \mathrm{i})$ $\mathrm{d}s^2 = \frac{1}{\cosh\rho}(\mathrm{d}\rho^2 - \sinh^2\rho\,\mathrm{d}\theta^2)$ $K = -\frac{\sinh^2\rho}{2\cosh\rho}$

classical limit $z \to 0$. Expressions (2.2) and (2.3) are explicitly written for each particular 'deformed' space in table 1; more details can be found in [12].

From (2.2) we compute the Christoffel Γ^i_{jk}, Riemann R^i_{jkl} and Ricci R_{ii} tensors [27]; their nonzero components turn out to be

$$\Gamma^\rho_{\rho\rho} = -\frac{1}{2}\lambda_1 \tanh(\lambda_1\rho) \qquad \Gamma^\theta_{\theta\rho} = \lambda_1 \frac{1+\cosh^2(\lambda_1\rho)}{\sinh(2\lambda_1\rho)}$$
$$\Gamma^\rho_{\theta\theta} = -\frac{\lambda_2^2}{2\lambda_1}\tanh(\lambda_1\rho)(1+\cosh^2(\lambda_1\rho)) \tag{2.4}$$

$$R^\rho_{\theta\rho\theta} = R_{\theta\theta} = -\frac{1}{2}\lambda_2^2 \sinh^2(\lambda_1\rho)\tanh^2(\lambda_1\rho) \qquad R^\theta_{\rho\theta\rho} = R_{\rho\rho} = -\frac{1}{2}\lambda_1^2\tanh^2(\lambda_1\rho). \tag{2.5}$$

The sectional and scalar curvatures are (2.3) and $2K(\rho)$, respectively. The connection (2.4) allows us to write the geodesic equations

$$\frac{\mathrm{d}^2 q_i}{\mathrm{d}s^2} + \Gamma^i_{jk}\frac{\mathrm{d}q_j}{\mathrm{d}s}\frac{\mathrm{d}q_k}{\mathrm{d}s} = 0 \tag{2.6}$$

which, in our case, give rise to the following equations for $(\rho(s), \theta(s))$ where s is the canonical parameter of the metric (2.2):

$$\frac{\mathrm{d}^2\rho}{\mathrm{d}s^2} - \frac{1}{2}\lambda_1\tanh(\lambda_1\rho)\left(\frac{\mathrm{d}\rho}{\mathrm{d}s}\right)^2 - \frac{\lambda_2^2}{2\lambda_1}\tanh(\lambda_1\rho)(1+\cosh^2(\lambda_1\rho))\left(\frac{\mathrm{d}\theta}{\mathrm{d}s}\right)^2 = 0 \tag{2.7}$$

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}s^2} + 2\lambda_1\left(\frac{1+\cosh^2(\lambda_1\rho)}{\sinh(2\lambda_1\rho)}\right)\frac{\mathrm{d}\rho}{\mathrm{d}s}\frac{\mathrm{d}\theta}{\mathrm{d}s} = 0. \tag{2.8}$$

The latter equation has a first integral given by

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\sinh^2(\lambda_1\rho)}{\lambda_1^2\cosh(\lambda_1\rho)}\frac{\mathrm{d}\theta}{\mathrm{d}s}\right) = 0 \quad\Longrightarrow\quad \frac{\sinh^2(\lambda_1\rho)}{\lambda_1^2\cosh(\lambda_1\rho)}\frac{\mathrm{d}\theta}{\mathrm{d}s} = -\alpha \tag{2.9}$$

where α is a constant. By substituting this result into the metric (2.2) we find the following velocities:

$$\left(\frac{\mathrm{d}\rho}{\mathrm{d}s}\right)^2 = \cosh(\lambda_1\rho) - \frac{\lambda_1^2\lambda_2^2\alpha^2}{\tanh^2(\lambda_1\rho)} \qquad \frac{\mathrm{d}\theta}{\mathrm{d}s} = -\alpha\frac{\lambda_1^2\cosh(\lambda_1\rho)}{\sinh^2(\lambda_1\rho)} \tag{2.10}$$

and equation (2.7) is fulfilled. Thus the geodesic curve $\rho = \rho(\theta)$ is obtained by eliminating the parameter s and then integrating the equation

$$\left(\frac{\mathrm{d}\rho}{\mathrm{d}\theta}\right)^2 = \frac{\sinh^2(\lambda_1\rho)}{\lambda_1^2\alpha^2}\left(\frac{\sinh^2(\lambda_1\rho)}{\lambda_1^2\cosh(\lambda_1\rho)} - \lambda_2^2\alpha^2\right) \tag{2.11}$$

which finally leads to an involved solution that depends on elliptic functions.

Note that all expressions (2.7)–(2.11) are well defined under the limit $\lambda_1 \to 0$. In this non-deformed/flat case we recover the known Euclidean and Minkowskian geodesics. Explicitly, from (2.10) and by taking the limit $\lambda_1 \to 0$, we find that

$$\rho^2 = s^2 + \lambda_2^2\alpha^2 \qquad \frac{\tan(\lambda_2(\theta+\theta_0))}{\lambda_2} = \frac{\alpha}{s} \tag{2.12}$$

where θ_0 is the second integration constant. Either from (2.12) or from (2.11) with $\lambda_1 = 0$, we obtain the curve

$$\frac{\alpha}{\rho} = \frac{\sin(\lambda_2(\theta+\theta_0))}{\lambda_2}. \tag{2.13}$$

2.2. *Superintegrable geodesic motion on spaces of constant curvature*

Let us now consider now the metric (1.13) that underlies the free superintegrable Hamiltonian (1.12). Under the change of coordinates (2.1) we find that

$$\mathrm{d}s^2 = \frac{1}{\cosh^2(\lambda_1\rho)}\left(\mathrm{d}\rho^2 + \lambda_2^2\frac{\sinh^2(\lambda_1\rho)}{\lambda_1^2}\,\mathrm{d}\theta^2\right). \tag{2.14}$$

By introducing a new radial coordinate r defined by [12]

$$r = \int_0^\rho \frac{\mathrm{d}x}{\cosh(\lambda_1 x)} \tag{2.15}$$

we obtain exactly the CK metric written in geodesic polar coordinates (r,θ) provided that now $\kappa_1 \equiv z = \lambda_1^2$ [26]:

$$\mathrm{d}s^2 = \mathrm{d}r^2 + \lambda_2^2\frac{\sin^2(\lambda_1 r)}{\lambda_1^2}\,\mathrm{d}\theta^2. \tag{2.16}$$

The nonzero connection and curvature tensors are given by

$$\Gamma^r_{\theta\theta} = -\lambda_2^2\frac{\sin(\lambda_1 r)}{\lambda_1}\cos(\lambda_1 r) \qquad \Gamma^\theta_{\theta r} = \frac{\lambda_1}{\tan(\lambda_1 r)} \tag{2.17}$$

$$R^r_{\theta r\theta} = R_{\theta\theta} = \lambda_2^2\sin^2(\lambda_1 r) \qquad R^\theta_{r\theta r} = R_{rr} = \lambda_1^2 \equiv z \tag{2.18}$$

so that the sectional curvature is just $K = z$. By applying the very same procedure described in section 2.1 we deduce the generic geodesic curve $r = r(\theta)$:

$$\frac{\lambda_1\alpha}{\tan(\lambda_1 r)} = \frac{\sin(\lambda_2(\theta+\theta_0))}{\lambda_2} \tag{2.19}$$

where α and θ_0 are the integration constants. In the flat case $z \to 0$ the coordinate $\rho \to r$, so that the curve (2.19) coincides with (2.13), as it should.

We specialize all this information for each space in table 2. When comparing tables 1 and 2, note that the sign of z for a given 'deformed' space of non-constant curvature is the opposite to the corresponding one of constant curvature (this is a consequence of definition (2.15)).

Table 2. Metric, connection, geodesics and sectional curvature of the six spaces of constant curvature for $\lambda_1 \in \{1, 0, \mathrm{i}\}$ and $\lambda_2 \in \{1, \mathrm{i}\}$ with deformed $sl(2)$-coalgebra symmetry.

2D Riemannian spaces	(1 + 1)D relativistic spacetimes
• Sphere $\mathbf{S}^2$: $(\lambda_1, \lambda_2) = (1, 1)$ $\mathrm{d}s^2 = \mathrm{d}r^2 + \sin^2 r\,\mathrm{d}\theta^2$ $\Gamma^r_{\theta\theta} = -\sin r \cos r \quad \Gamma^\theta_{\theta r} = 1/\tan r$ $\alpha/\tan r = \sin(\theta + \theta_0)$ $K = 1$	• Anti-de Sitter spacetime $\mathbf{AdS}^{1+1}$: $(\lambda_1, \lambda_2) = (1, \mathrm{i})$ $\mathrm{d}s^2 = \mathrm{d}r^2 - \sin^2 r\,\mathrm{d}\theta^2$ $\Gamma^r_{\theta\theta} = \sin r \cos r \quad \Gamma^\theta_{\theta r} = 1/\tan r$ $\alpha/\tan r = \sinh(\theta + \theta_0)$ $K = 1$
• Euclidean space $\mathbf{E}^2$: $(\lambda_1, \lambda_2) = (0, 1)$ $\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2$ $\Gamma^r_{\theta\theta} = -r \quad \Gamma^\theta_{\theta r} = 1/r$ $\alpha/r = \sin(\theta + \theta_0)$ $K = 0$	• Minkowskian spacetime $\mathbf{M}^{1+1}$: $(\lambda_1, \lambda_2) = (0, \mathrm{i})$ $\mathrm{d}s^2 = \mathrm{d}r^2 - r^2\,\mathrm{d}\theta^2$ $\Gamma^r_{\theta\theta} = r \quad \Gamma^\theta_{\theta r} = 1/r$ $\alpha/r = \sinh(\theta + \theta_0)$ $K = 0$
• Hyperbolic space $\mathbf{H}^2$: $(\lambda_1, \lambda_2) = (\mathrm{i}, 1)$ $\mathrm{d}s^2 = \mathrm{d}r^2 + \sinh^2 r\,\mathrm{d}\theta^2$ $\Gamma^r_{\theta\theta} = -\sinh r \cosh r \quad \Gamma^\theta_{\theta r} = 1/\tanh r$ $\alpha/\tanh r = \sin(\theta + \theta_0)$ $K = -1$	• De Sitter spacetime $\mathbf{dS}^{1+1}$: $(\lambda_1, \lambda_2) = (\mathrm{i}, \mathrm{i})$ $\mathrm{d}s^2 = \mathrm{d}r^2 - \sinh^2 r\,\mathrm{d}\theta^2$ $\Gamma^r_{\theta\theta} = \sinh r \cosh r \quad \Gamma^\theta_{\theta r} = 1/\tanh r$ $\alpha/\tanh r = \sinh(\theta + \theta_0)$ $K = -1$

3. Integrable potentials on spaces of non-constant curvature

Let (p_ρ, p_θ) be the canonical momenta corresponding to the new coordinates (ρ, θ) (2.1). The relationship between the initial phase space $(q_1, q_2; p_1, p_2)$ and the new one $(\rho, \theta; p_\rho, p_\theta)$ is found to be

$$\begin{aligned} \frac{p_1}{q_1} &= \frac{\lambda_1}{\tanh(\lambda_1\rho)} p_\rho + \frac{\lambda_1^2}{\lambda_2 \sinh^2(\lambda_1\rho)\tan(\lambda_2\theta)} p_\theta \\ \frac{p_2}{q_2} &= \frac{\lambda_1}{\tanh(\lambda_1\rho)} p_\rho - \frac{\lambda_1^2 \tan(\lambda_2\theta)}{\lambda_2 \tanh^2(\lambda_1\rho)} p_\theta. \end{aligned} \tag{3.1}$$

Therefore if we consider the generic integrable Hamiltonian

$$\mathcal{H}_z^{\mathrm{I}} = \tfrac{1}{2} J_+ + \mathcal{U}(zJ_-) \tag{3.2}$$

and we perform the corresponding transformations we get

$$\begin{aligned} H_z^{\mathrm{I}} = \frac{1}{2}\cosh(\lambda_1\rho)\left(p_\rho^2 + \frac{\lambda_1^2}{\lambda_2^2 \sinh^2(\lambda_1\rho)} p_\theta^2\right) + g(\rho) \\ + \frac{2\lambda_1^2\cosh(\lambda_1\rho)}{\sinh^2(\lambda_1\rho)}\left(\frac{b_1}{\sin^2(\lambda_2\theta)} + \frac{b_2}{\cos^2(\lambda_2\theta)}\right) \end{aligned} \tag{3.3}$$

where $H_z^{\mathrm{I}} = 2\mathcal{H}_z^{\mathrm{I}}$ and $g(\rho) = 2\mathcal{U}(zJ_-(\rho))$ is an arbitrary smooth function. The corresponding constant of the motion follows from (1.6). If we define $C_z(\rho, \theta; p_\rho, p_\theta) = 4\lambda_2^2 \mathcal{C}_z(q_i, p_i)$ we find that

$$C_z = p_\theta^2 + \frac{4\lambda_2^2 b_1}{\sin^2(\lambda_2\theta)} + \frac{4\lambda_2^2 b_2}{\cos^2(\lambda_2\theta)} \tag{3.4}$$

which does not depend on (ρ, p_ρ). Furthermore this constant allows us to reduce (3.3) to the 1D (radial) Hamiltonian given by

$$H_z^{\mathrm{I}}(\rho, p_\rho) = \frac{1}{2}\cosh(\lambda_1\rho) p_\rho^2 + \frac{\lambda_1^2\cosh(\lambda_1\rho)}{2\lambda_2^2\sinh^2(\lambda_1\rho)} C_z + g(\rho). \tag{3.5}$$

Table 3. Integrable Hamiltonians and their constant of the motion on the six spaces given in table 1. The same conventions are followed.

• Deformed sphere $\mathbf{S}_z^2$	• Deformed anti-de Sitter spacetime $\mathbf{AdS}_z^{1+1}$
$H_z^{\mathrm{I}} = \frac{1}{2}\cos\rho\left(p_\rho^2 + \frac{1}{\sin^2\rho}p_\theta^2\right) + g(\rho)$	$H_z^{\mathrm{I}} = \frac{1}{2}\cos\rho\left(p_\rho^2 - \frac{1}{\sin^2\rho}p_\theta^2\right) + g(\rho)$
$\quad + \frac{2\cos\rho}{\sin^2\rho}\left(\frac{b_1}{\sin^2\theta} + \frac{b_2}{\cos^2\theta}\right)$	$\quad - \frac{2\cos\rho}{\sin^2\rho}\left(\frac{b_1}{\sinh^2\theta} - \frac{b_2}{\cosh^2\theta}\right)$
$\quad = \frac{1}{2}\cos\rho p_\rho^2 + \frac{\cos\rho}{2\sin^2\rho}C_z + g(\rho)$	$\quad = \frac{1}{2}\cos\rho p_\rho^2 - \frac{\cos\rho}{2\sin^2\rho}C_z + g(\rho)$
$C_z = p_\theta^2 + \frac{4b_1}{\sin^2\theta} + \frac{4b_2}{\cos^2\theta}$	$C_z = p_\theta^2 + \frac{4b_1}{\sinh^2\theta} - \frac{4b_2}{\cosh^2\theta}$
• Euclidean space $\mathbf{E}^2$	• Minkowskian spacetime $\mathbf{M}^{1+1}$
$H^{\mathrm{I}} = \frac{1}{2}\left(p_\rho^2 + \frac{1}{\rho^2}p_\theta^2\right) + g(\rho)$	$H^{\mathrm{I}} = \frac{1}{2}\left(p_\rho^2 - \frac{1}{\rho^2}p_\theta^2\right) + g(\rho)$
$\quad + \frac{2}{\rho^2}\left(\frac{b_1}{\sin^2\theta} + \frac{b_2}{\cos^2\theta}\right)$	$\quad - \frac{2}{\rho^2}\left(\frac{b_1}{\sinh^2\theta} - \frac{b_2}{\cosh^2\theta}\right)$
$\quad = \frac{1}{2}p_\rho^2 + \frac{1}{2\rho^2}C_z + g(\rho)$	$\quad = \frac{1}{2}p_\rho^2 - \frac{1}{2\rho^2}C_z + g(\rho)$
$C = p_\theta^2 + \frac{4b_1}{\sin^2\theta} + \frac{4b_2}{\cos^2\theta}$	$C = p_\theta^2 + \frac{4b_1}{\sinh^2\theta} - \frac{4b_2}{\cosh^2\theta}$
• Deformed hyperbolic space $\mathbf{H}_z^2$	• Deformed de Sitter spacetime $\mathbf{dS}_z^{1+1}$
$H_z^{\mathrm{I}} = \frac{1}{2}\cosh\rho\left(p_\rho^2 + \frac{1}{\sinh^2\rho}p_\theta^2\right) + g(\rho)$	$H_z^{\mathrm{I}} = \frac{1}{2}\cosh\rho\left(p_\rho^2 - \frac{1}{\sinh^2\rho}p_\theta^2\right) + g(\rho)$
$\quad + \frac{2\cosh\rho}{\sinh^2\rho}\left(\frac{b_1}{\sin^2\theta} + \frac{b_2}{\cos^2\theta}\right)$	$\quad - \frac{2\cosh\rho}{\sinh^2\rho}\left(\frac{b_1}{\sinh^2\theta} - \frac{b_2}{\cosh^2\theta}\right)$
$\quad = \frac{1}{2}\cosh\rho p_\rho^2 + \frac{\cosh\rho}{2\sinh^2\rho}C_z + g(\rho)$	$\quad = \frac{1}{2}\cosh\rho p_\rho^2 - \frac{\cosh\rho}{2\sinh^2\rho}C_z + g(\rho)$
$C_z = p_\theta^2 + \frac{4b_1}{\sin^2\theta} + \frac{4b_2}{\cos^2\theta}$	$C_z = p_\theta^2 + \frac{4b_1}{\sinh^2\theta} - \frac{4b_2}{\cosh^2\theta}$

Consequently, the Hamiltonian (3.3) defines a family of integrable systems which, for any $g(\rho)$, share the same constant of the motion (3.4). We specialize in table 3 these expressions for each of the six spaces shown in table 1.

3.1. The SW potential

Amongst the possible choices for the deformed Hamiltonian (3.2), let us consider the expression (1.16) whose non-deformed limit is the SW system (1.14). This choice implies that the potential function $g(\rho)$ appearing in (3.3) reads

$$g(\rho) = \beta_0 \cosh(\lambda_1\rho)\frac{\tanh^2(\lambda_1\rho)}{\lambda_1^2}. \tag{3.6}$$

This gives

$$H_z^{\mathrm{ISW}} = \cosh(\lambda_1\rho)H_z^{\mathrm{SW}} \tag{3.7}$$

where

$$\begin{aligned} H_z^{\mathrm{SW}} = \frac{1}{2}\left(p_\rho^2 + \frac{\lambda_1^2}{\lambda_2^2\sinh^2(\lambda_1\rho)}p_\theta^2\right) + \beta_0\frac{\tanh^2(\lambda_1\rho)}{\lambda_1^2} \\ + \frac{2\lambda_1^2}{\sinh^2(\lambda_1\rho)}\left(\frac{b_1}{\sin^2(\lambda_2\theta)} + \frac{b_2}{\cos^2(\lambda_2\theta)}\right). \end{aligned} \tag{3.8}$$

The physical interpretation of (3.7) is the following.

- When $z = 0$ ($\lambda_1 \to 0$), expression (3.7) reduces to the SW Hamiltonian on $\mathbf{E}^2$ and $\mathbf{M}^{1+1}$ written in polar coordinates

$$H^{\mathrm{ISW}} \equiv H^{\mathrm{SW}} = \frac{1}{2}\left(p_\rho^2 + \frac{p_\theta^2}{\lambda_2^2\rho^2}\right) + \beta_0\rho^2 + \frac{2}{\rho^2}\left(\frac{b_1}{\sin^2(\lambda_2\theta)} + \frac{b_2}{\cos^2(\lambda_2\theta)}\right). \tag{3.9}$$

The term $\beta_0\rho^2$ is the usual harmonic oscillator potential, while those depending on b_1 and b_2 are two 'centrifugal barriers'.

- When $z \neq 0$, $H_z^{\rm SW}$ is well known in the literature as the superintegrable SW system on spaces of constant curvature [19–23], where the β_0-term corresponds to a 'curved' harmonic oscillator potential. In particular, if $z = -1$ we recover the spherical oscillator or Higgs potential [28, 29], $\beta_0 \tan^2\rho$, on $\mathbf{S}^2$ and $\mathbf{AdS}^{1+1}$, while $z = 1$ leads to a hyperbolic oscillator $\beta_0 \tanh^2\rho$ on $\mathbf{H}^2$ and $\mathbf{dS}^{1+1}$. Hence $H_z^{\rm ISW}$ can properly be regarded as an integrable generalization of the superintegrable $H_z^{\rm SW}$ to spaces of non-constant curvature. We stress that $H_z^{\rm SW}$ can be reproduced from the Hamiltonian (1.18) in an exact way and it will be analysed in the following section.

3.2. The KC potential

If we now consider the Hamiltonian (1.17) whose non-deformed limit is the KC system (1.15), we get

$$g(\rho) = -k\cosh(\lambda_1\rho)\frac{\lambda_1}{\tanh(\lambda_1\rho)} \tag{3.10}$$

with $k = 2\sqrt{2}\gamma$. With this choice we find that

$$H_z^{\rm IKC} = \cosh(\lambda_1\rho) H_z^{\rm KC} \tag{3.11}$$

where

$$H_z^{\rm KC} = \frac{1}{2}\left(p_\rho^2 + \frac{\lambda_1^2}{\lambda_2^2\sinh^2(\lambda_1\rho)}p_\theta^2\right) - k\frac{\lambda_1}{\tanh(\lambda_1\rho)} + \frac{2\lambda_1^2}{\sinh^2(\lambda_1\rho)}\left(\frac{b_1}{\sin^2(\lambda_2\theta)} + \frac{b_2}{\cos^2(\lambda_2\theta)}\right) \tag{3.12}$$

and the integrable (but not superintegrable) Hamiltonian $H_z^{\rm KC}$ contains a KC potential in polar coordinates through the k-term. Explicitly,

- When $z = 0$ we obtain the reduction
$$H^{\rm IKC} \equiv H^{\rm KC} = \frac{1}{2}\left(p_\rho^2 + \frac{p_\theta^2}{\lambda_2^2\rho^2}\right) - \frac{k}{\rho} + \frac{2}{\rho^2}\left(\frac{b_1}{\sin^2(\lambda_2\theta)} + \frac{b_2}{\cos^2(\lambda_2\theta)}\right) \tag{3.13}$$
which defines an integrable system formed by a composition of the KC potential, $-k/\rho$, with two centrifugal barriers on the flat spaces $\mathbf{E}^2$ and $\mathbf{M}^{1+1}$. This Hamiltonian is superintegrable whenever at least one of the parameters b_1, b_2 is taken equal to zero (see, e.g., [21, 30]).
- In contrast, if $z \neq 0$ then $H_z^{\rm IKC} \neq H_z^{\rm KC}$ and the latter contains the KC potential on spaces of constant curvature [21–23, 31–33] in its spherical version, $-k/\tan\rho$, on $\mathbf{S}^2$ [34] and $\mathbf{AdS}^{1+1}(z = -1)$, as well as in its hyperbolic one, $-k/\tanh\rho$, on $\mathbf{H}^2$ and $\mathbf{dS}^{1+1}(z = 1)$. Therefore we can conclude that $H_z^{\rm IKC}$ (3.11) is an appropriate generalization of the KC potential to spaces of non-constant curvature, possibly supplemented by two more potential terms.

4. The superintegrable SW potential on spaces with constant curvature

Now we can reproduce the same study in the case of the superintegrable Hamiltonian (1.18) defined on the spaces of constant curvature of section 2.2.

The relation between the initial phase space (q_i, p_i) and the canonical geodesic polar coordinates (r, θ) and momenta (p_r, p_θ) turns out to be

$$\begin{aligned} \frac{p_1}{q_1} &= \frac{\lambda_1}{\tan(\lambda_1 r)} p_r + \frac{\lambda_1^2}{\lambda_2 \tan^2(\lambda_1 r)\tan(\lambda_2\theta)} p_\theta \\ \frac{p_2}{q_2} &= \frac{\lambda_1}{\tan(\lambda_1 r)} p_r - \frac{\lambda_1^2 \tan(\lambda_2\theta)}{\lambda_2 \sin^2(\lambda_1 r)} p_\theta . \end{aligned} \tag{4.1}$$

By considering the realization (1.5), the change of coordinates defined through the composition of (2.1) and (2.15) together with the above relations, it can be shown that the Hamiltonian $\mathcal{H}_z^{\mathrm{SSW}}$ (1.18) and its constants of the motion $\mathcal{C}_z$ (1.6) and $\mathcal{I}_z$ (1.19) are expressed in the phase space $(r, \theta; p_r, p_\theta)$ as

$$H_z^{\mathrm{SW}} = \frac{1}{2}\left(p_r^2 + \frac{\lambda_1^2}{\lambda_2^2 \sin^2(\lambda_1 r)} p_\theta^2\right) + \beta_0 \frac{\tan^2(\lambda_1 r)}{\lambda_1^2} + \frac{\lambda_1^2}{\sin^2(\lambda_1 r)}\left(\frac{\beta_1}{\cos^2(\lambda_2\theta)} + \frac{\beta_2}{\sin^2(\lambda_2\theta)}\right) \tag{4.2}$$

$$C_z = p_\theta^2 + \frac{2\lambda_2^2\beta_1}{\cos^2(\lambda_2\theta)} + \frac{2\lambda_2^2\beta_2}{\sin^2(\lambda_2\theta)} \tag{4.3}$$

$$\begin{aligned} I_z = {} & \left(\lambda_2 \sin(\lambda_2\theta) p_r + \frac{\lambda_1 \cos(\lambda_2\theta)}{\tan(\lambda_1 r)} p_\theta\right)^2 \\ & + 2\beta_0\lambda_2^2 \frac{\tan^2(\lambda_1 r)}{\lambda_1^2}\sin^2(\lambda_2\theta) + \frac{2\beta_2\lambda_1^2\lambda_2^2}{\tan^2(\lambda_1 r)\sin^2(\lambda_2\theta)} \end{aligned} \tag{4.4}$$

where we have rescaled these quantities in the following way:

$$\begin{aligned} & H_z^{\mathrm{SW}} = 2\mathcal{H}_z^{\mathrm{SSW}} \qquad C_z = 4\lambda_2^2 \mathcal{C}_z \qquad I_z = 4\lambda_2^2\left(\mathcal{I}_z - \frac{\beta_0}{2\lambda_1^2} - \lambda_1^2 b_1\right) \\ & \beta_1 = 2b_2 \qquad \beta_2 = 2b_1 . \end{aligned} \tag{4.5}$$

The constant of the motion C_z allows us to reduce H_z^{SW} to a 1D radial system given by

$$H_z^{\mathrm{SW}} = \frac{1}{2} p_r^2 + \frac{\lambda_1^2}{2\lambda_2^2 \sin^2(\lambda_1 r)} C_z + \beta_0 \frac{\tan^2(\lambda_1 r)}{\lambda_1^2} . \tag{4.6}$$

These results are displayed in table 4 for each space of constant curvature. The superintegrable Hamiltonians written in the first column of table 4 are constructed on the three classical *Riemannian spaces* ($\lambda_2 = 1$) and they are already known [19–23]. We recall that all these results were obtained by applying different procedures that are not related to coalgebra symmetry. In geodesic polar coordinates these systems were also constructed in [35, 36] through a Lie group approach, where the constants of the motion there denoted by I_{12}, I_{02} are related to (4.3) and (4.4) by $I_{12} = C_z - 2\lambda_2^2(\beta_1 + \beta_2)$, $I_{02} = I_z$. The Hamiltonian H_z^{SW} on $\mathbf{S}^2$ has been interpreted in [37–40] as a superposition of three spherical oscillators; the corresponding geometrical interpretation on $\mathbf{H}^2$ can be found in [36].

We stress that the second column of table 4 provides the coalgebraic generalization of the SW potential to the three 'classical' *relativistic spacetimes* ($\lambda_2 = \mathrm{i}$) which, to our knowledge, was still lacking. In the following we shall describe the system H_z^{SW} on the six spaces, that includes a full description of the new cases (on $\mathbf{AdS}^{1+1}$, $\mathbf{M}^{1+1}$ and $\mathbf{dS}^{1+1}$), as well as of the known potentials on $\mathbf{S}^2$, $\mathbf{E}^2$ and $\mathbf{H}^2$. We point out that when the six spaces are considered altogether the interpretation becomes more comprehensible and transparent.

Table 4. Superintegrable SW Hamiltonians and their two constants of the motion on the six spaces of constant curvature of table 2 and with the same conventions.

• Sphere $\mathbf{S}^2$ $\begin{aligned} H_z^{\mathrm{SW}} &= \tfrac{1}{2}\left(p_r^2 + \tfrac{1}{\sin^2 r} p_\theta^2\right) + \beta_0 \tan^2 r \\ &\quad + \tfrac{1}{\sin^2 r}\left(\tfrac{\beta_1}{\cos^2\theta} + \tfrac{\beta_2}{\sin^2\theta}\right) \\ &= \tfrac{1}{2} p_r^2 + \tfrac{1}{2\sin^2 r} C_z + \beta_0 \tan^2 r \\ C_z &= p_\theta^2 + \tfrac{2\beta_1}{\cos^2\theta} + \tfrac{2\beta_2}{\sin^2\theta} \\ I_z &= \left(\sin\theta p_r + \tfrac{\cos\theta}{\tan r} p_\theta\right)^2 \\ &\quad + 2\beta_0 \tan^2 r \sin^2\theta + \tfrac{2\beta_2}{\tan^2 r \sin^2\theta} \end{aligned}$	• Anti-de Sitter spacetime $\mathbf{AdS}^{1+1}$ $\begin{aligned} H_z^{\mathrm{SW}} &= \tfrac{1}{2}\left(p_r^2 - \tfrac{1}{\sin^2 r} p_\theta^2\right) + \beta_0 \tan^2 r \\ &\quad + \tfrac{1}{\sin^2 r}\left(\tfrac{\beta_1}{\cosh^2\theta} - \tfrac{\beta_2}{\sinh^2\theta}\right) \\ &= \tfrac{1}{2} p_r^2 - \tfrac{1}{2\sin^2 r} C_z + \beta_0 \tan^2 r \\ C_z &= p_\theta^2 - \tfrac{2\beta_1}{\cosh^2\theta} + \tfrac{2\beta_2}{\sinh^2\theta} \\ I_z &= \left(\sinh\theta p_r - \tfrac{\cosh\theta}{\tan r} p_\theta\right)^2 \\ &\quad + 2\beta_0 \tan^2 r \sinh^2\theta + \tfrac{2\beta_2}{\tan^2 r \sinh^2\theta} \end{aligned}$
• Euclidean space $\mathbf{E}^2$ $\begin{aligned} H^{\mathrm{SW}} &= \tfrac{1}{2}\left(p_r^2 + \tfrac{1}{r^2} p_\theta^2\right) + \beta_0 r^2 \\ &\quad + \tfrac{1}{r^2}\left(\tfrac{\beta_1}{\cos^2\theta} + \tfrac{\beta_2}{\sin^2\theta}\right) \\ &= \tfrac{1}{2} p_r^2 + \tfrac{1}{2r^2} C_z + \beta_0 r^2 \\ C &= p_\theta^2 + \tfrac{2\beta_1}{\cos^2\theta} + \tfrac{2\beta_2}{\sin^2\theta} \\ I &= \left(\sin\theta\, p_r + \tfrac{\cos\theta}{r} p_\theta\right)^2 \\ &\quad + 2\beta_0 r^2 \sin^2\theta + \tfrac{2\beta_2}{r^2 \sin^2\theta} \end{aligned}$	• Minkowskian spacetime $\mathbf{M}^{1+1}$ $\begin{aligned} H^{\mathrm{SW}} &= \tfrac{1}{2}\left(p_r^2 - \tfrac{1}{r^2} p_\theta^2\right) + \beta_0 r^2 \\ &\quad + \tfrac{1}{r^2}\left(\tfrac{\beta_1}{\cosh^2\theta} - \tfrac{\beta_2}{\sinh^2\theta}\right) \\ &= \tfrac{1}{2} p_r^2 - \tfrac{1}{2r^2} C_z + \beta_0 r^2 \\ C &= p_\theta^2 - \tfrac{2\beta_1}{\cosh^2\theta} + \tfrac{2\beta_2}{\sinh^2\theta} \\ I &= \left(\sinh\theta\, p_r - \tfrac{\cosh\theta}{r} p_\theta\right)^2 \\ &\quad + 2\beta_0 r^2 \sinh^2\theta + \tfrac{2\beta_2}{r^2 \sinh^2\theta} \end{aligned}$
• Hyperbolic space $\mathbf{H}^2$ $\begin{aligned} H_z^{\mathrm{SW}} &= \tfrac{1}{2}\left(p_r^2 + \tfrac{1}{\sinh^2 r} p_\theta^2\right) + \beta_0 \tanh^2 r \\ &\quad + \tfrac{1}{\sinh^2 r}\left(\tfrac{\beta_1}{\cos^2\theta} + \tfrac{\beta_2}{\sin^2\theta}\right) \\ &= \tfrac{1}{2} p_r^2 + \tfrac{1}{2\sinh^2 r} C_z + \beta_0 \tanh^2 r \\ C_z &= p_\theta^2 + \tfrac{2\beta_1}{\cos^2\theta} + \tfrac{2\beta_2}{\sin^2\theta} \\ I_z &= \left(\sin\theta\, p_r + \tfrac{\cos\theta}{\tanh r} p_\theta\right)^2 \\ &\quad + 2\beta_0 \tanh^2 r \sin^2\theta + \tfrac{2\beta_2}{\tanh^2 r \sin^2\theta} \end{aligned}$	• De Sitter spacetime $\mathbf{dS}^{1+1}$ $\begin{aligned} H_z^{\mathrm{SW}} &= \tfrac{1}{2}\left(p_r^2 - \tfrac{1}{\sinh^2 r} p_\theta^2\right) + \beta_0 \tanh^2 r \\ &\quad + \tfrac{1}{\sinh^2 r}\left(\tfrac{\beta_1}{\cosh^2\theta} - \tfrac{\beta_2}{\sinh^2\theta}\right) \\ &= \tfrac{1}{2} p_r^2 - \tfrac{1}{2\sinh^2 r} C_z + \beta_0 \tanh^2 r \\ C_z &= p_\theta^2 - \tfrac{2\beta_1}{\cosh^2\theta} + \tfrac{2\beta_2}{\sinh^2\theta} \\ I_z &= \left(\sinh\theta\, p_r - \tfrac{\cosh\theta}{\tanh r} p_\theta\right)^2 \\ &\quad + 2\beta_0 \tanh^2 r \sinh^2\theta + \tfrac{2\beta_2}{\tanh^2 r \sinh^2\theta} \end{aligned}$

4.1. Geometrical interpretation of the SW potential

Let us firstly recall the (physical) geometrical role of the geodesic polar coordinates (r, θ) on the spaces of constant curvature of table 2. These can be embedded in a 3D linear ambient space with coordinates (x_0, x_1, x_2) given by [26]

$$x_0 = \cos(\lambda_1 r) \qquad x_1 = \frac{\sin(\lambda_1 r)}{\lambda_1}\cos(\lambda_2\theta) \qquad x_2 = \frac{\sin(\lambda_1 r)}{\lambda_1}\frac{\sin(\lambda_2\theta)}{\lambda_2} \tag{4.7}$$

which fulfil the constraint $x_0^2 + \lambda_1^2\left(x_1^2 + \lambda_2^2 x_2^2\right) = 1$; the origin has ambient coordinates $O = (1, 0, 0)$. Consider a (time-like) geodesic l_1, another (space-like) geodesic l_2 orthogonal at O and a generic point Q with coordinates (4.7). Then r is the (time-like) distance along the geodesic l that joins Q with O, while θ is the (rapidity) angle of l with respect to l_1 as shown in figure 1. In the Riemannian cases with $\lambda_2 = 1$ the coordinates (r, θ) parametrize the complete space, while in the relativistic cases with $\lambda_2 = \mathrm{i}$, it is verified that $|x_2| \leqslant |x_1|$, so that (r, θ) only cover the time-like lines limited by the isotropic lines $x_2 = \pm x_1$ on which $\theta \to \infty$.

Next, let Q_i $(i = 1, 2)$ be the intersection point of l_i with its orthogonal geodesic through Q. Then if x denotes the (time-like) distance QQ_2 and y the (space-like) distance QQ_1, it can be shown that [26]

$$x_1 = \frac{\sin(\lambda_1 r)}{\lambda_1}\cos(\lambda_2\theta) \equiv \frac{\sin(\lambda_1 x)}{\lambda_1} \qquad x_2 = \frac{\sin(\lambda_1 r)}{\lambda_1}\frac{\sin(\lambda_2\theta)}{\lambda_2} \equiv \frac{\sin(\lambda_1\lambda_2 y)}{\lambda_1\lambda_2}. \tag{4.8}$$

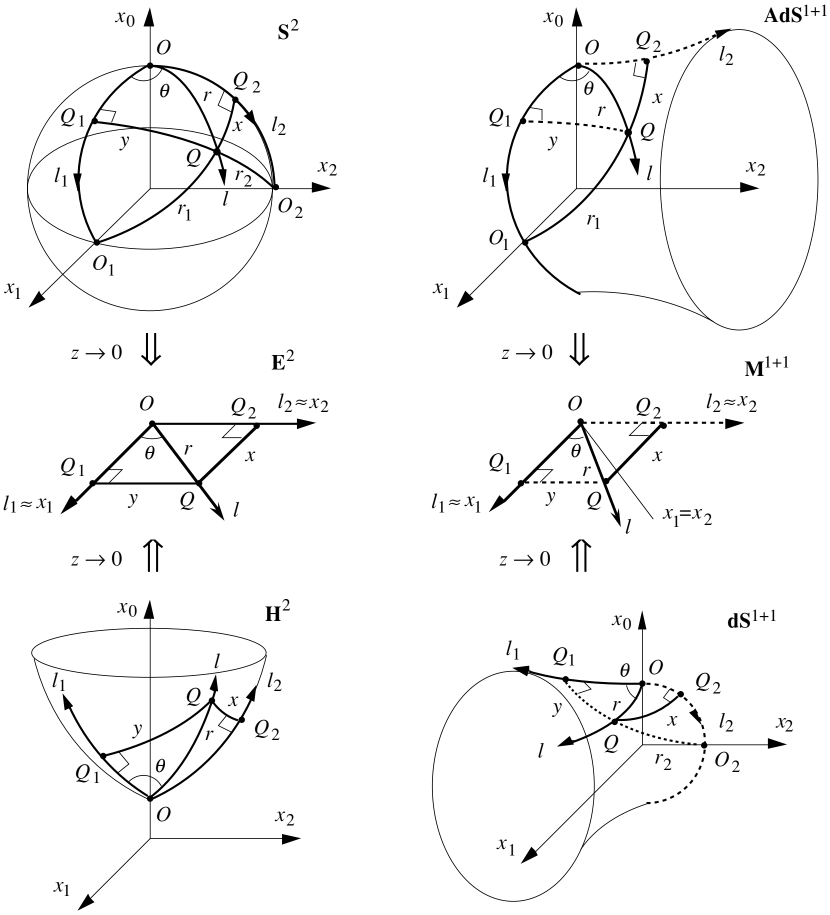

Figure 1. Geometrical description of the SW potential on the six spaces of tables 2 and 4 in a 3D linear ambient space. In the three spacetimes (right) space-like lines/distances are represented by dashed lines.

Hence the potential of $H_z^{\rm SW}$ (4.2) can be rewritten as

$$U_z^{\rm SW} = \beta_0 \frac{\tan^2(\lambda_1 r)}{\lambda_1^2} + \frac{\lambda_1^2 \beta_1}{\sin^2(\lambda_1 x)} + \frac{\lambda_1^2 \beta_2}{\sin^2(\lambda_1 \lambda_2 y)} \tag{4.9}$$

which conveys a common interpretation on the six spaces:

- the β_0-term is a *central* harmonic oscillator, that is, with centre at the origin O;
- both β_i-terms ($i = 1, 2$) are 'centrifugal barriers'.

Alternatively, the β_i-terms may adopt a different interpretation in each particular space, depicted in figure 1, that we proceed to study. All the trigonometric relations on these spaces that are necessary in our description can be found in [25].

4.1.1. Sphere $\mathbf{S}^2$. Let $O_i (i = 1, 2)$ be the intersection points of the geodesics l_i with the equator of the sphere (at a distance $\frac{\pi}{2}$ from O) and r_i the distances along the geodesics joining

Q and O_i. By applying the cosine theorem for the two triangles OQO_i we find that

$$\begin{aligned}\cos r_1 &= \cos\frac{\pi}{2}\cos r + \sin\frac{\pi}{2}\sin r\cos\theta \\ \cos r_2 &= \cos\frac{\pi}{2}\cos r + \sin\frac{\pi}{2}\sin r\cos\left(\frac{\pi}{2}-\theta\right)\end{aligned} \tag{4.10}$$

that is

$$\cos r_1 = \sin r\cos\theta \equiv \sin x \qquad \cos r_2 = \sin r\sin\theta \equiv \sin y. \tag{4.11}$$

The same result follows by noting that $r_1 + x = r_2 + y = \frac{\pi}{2}$ and that $\{O_1, Q, Q_2\}$ and $\{O_2, Q, Q_1\}$ lie on the same geodesic orthogonal to l_2 and l_1, respectively.

Therefore the potential (4.9) on $\mathbf{S}^2$ can be expressed in two ways

$$U_z^{\mathrm{SW}} = \beta_0\tan^2 r + \frac{\beta_1}{\sin^2 x} + \frac{\beta_2}{\sin^2 y} \tag{4.12}$$

$$= \beta_0\tan^2 r + \beta_1\tan^2 r_1 + \beta_2\tan^2 r_2 + \beta_1 + \beta_2 \tag{4.13}$$

which show a superposition of the central spherical oscillator either with two spherical centrifugal barriers, or with two spherical oscillators with centres placed at O_i [35–40].

4.1.2. Hyperbolic space $\mathbf{H}^2$. In this case, the analogous points to the previous 'centres' O_i would be beyond the 'actual' hyperbolic space and so placed in the 'ideal' one [37–39] (in the exterior region of $\mathbf{H}^2$). Thus we only write the potential in the form (4.9),

$$U_z^{\mathrm{SW}} = \beta_0\tanh^2 r + \frac{\beta_1}{\sinh^2 x} + \frac{\beta_2}{\sinh^2 y} \tag{4.14}$$

which corresponds to a superposition of a hyperbolic oscillator centred at the origin with two hyperbolic 'centrifugal barriers' [36].

4.1.3. Euclidean space $\mathbf{E}^2$. The limit $z \to 0$ of the potential U_z^{SW} on $\mathbf{S}^2$ and $\mathbf{H}^2$ can properly be performed on both expressions (4.12) and (4.14) leading to the usual harmonic oscillator $\beta_0 r^2 = \beta_0(x^2+y^2)$ and centrifugal potentials β_1/x^2 and β_2/y^2, in such a manner that (x, y) are Cartesian coordinates on $\mathbf{E}^2$. Thus the proper SW system is recovered [14–17]. Nevertheless, this contraction cannot directly be applied to the potential on $\mathbf{S}^2$ written in terms of r_i (4.13); in fact, when $z \to 0$ the points $O_i \to \infty$.

4.1.4. Anti-de Sitter spacetime $\mathbf{AdS}^{1+1}$. We consider the intersection point O_1 between the time-like geodesic l_1 and the axis x_1 of the ambient space, which is at a time-like distance $\frac{\pi}{2}$ from the origin O. If r_1 denotes the time-like distance QO_1, by applying the cosine theorem on the triangle OQO_1, we find that

$$\cos r_1 = \cos\frac{\pi}{2}\cos r + \sin\frac{\pi}{2}\sin r\cosh\theta \quad\Longrightarrow\quad \cos r_1 = \sin r\cosh\theta \equiv \sin x, \tag{4.15}$$

that is, $r_1 + x = \frac{\pi}{2}$. Therefore the potential becomes

$$U_z^{\mathrm{SW}} = \beta_0\tan^2 r + \frac{\beta_1}{\sin^2 x} - \frac{\beta_2}{\sinh^2 y} \tag{4.16}$$

$$= \beta_0\tan^2 r + \beta_1\tan^2 r_1 - \frac{\beta_2}{\sinh^2 y} + \beta_1. \tag{4.17}$$

The former expression exhibits a superposition of a time-like (spherical) oscillator centred at O with a time-like (spherical) centrifugal potential and a space-like (hyperbolic) one. Under the latter form, the time-like centrifugal term is transformed into another spherical oscillator now with centre at O_1.

4.1.5. De Sitter spacetime $\mathbf{dS}^{1+1}$. Recall that $\mathbf{AdS}^{1+1}$ and $\mathbf{dS}^{1+1}$ are related by means of an interchange between time-like lines and space-like ones; the former are compact (circular) on $\mathbf{AdS}^{1+1}$ and non-compact (hyperbolic) on $\mathbf{dS}^{1+1}$, while the latter are non-compact on $\mathbf{AdS}^{1+1}$ but compact on $\mathbf{dS}^{1+1}$. So, conversely to the previous case, we consider the intersection point O_2 between the space-like geodesic l_2 and the axis x_2 (at a space-like distance $\frac{\pi}{2}$ from O), such that r_2 is the space-like distance QO_2. The cosine theorem applied to the triangle OQQ_2 (with external angle θ) gives rise to

$$\cos r_2 = \cos\frac{\pi}{2}\cosh r + \sin\frac{\pi}{2}\sinh r\sinh\theta \quad\Longrightarrow\quad \cos r_2 = \sinh r\sinh\theta \equiv \sin y. \tag{4.18}$$

Note that $r_2 + y = \frac{\pi}{2}$. Hence we again find two ways to express the potential on $\mathbf{dS}^{1+1}$

$$U_z^{\mathrm{SW}} = \beta_0\tanh^2 r + \frac{\beta_1}{\sinh^2 x} - \frac{\beta_2}{\sin^2 y} \tag{4.19}$$

$$= \beta_0\tanh^2 r + \frac{\beta_1}{\sinh^2 x} - \beta_2\tan^2 r_2 - \beta_2 \tag{4.20}$$

that is, a superposition of a central time-like (hyperbolic) oscillator with a time-like (hyperbolic) centrifugal barrier, and either with another space-like (spherical) centrifugal barrier or with a space-like (spherical) oscillator centred at O_2.

4.1.6. Minkowskian spacetime $\mathbf{M}^{1+1}$. The limit $z\to 0$ of (4.16) and (4.19) provides the corresponding SW potential on $\mathbf{M}^{1+1}$, which is formed by a time-like harmonic oscillator $\beta_0 r^2 = \beta_0(x^2 - y^2)$, one time-like centrifugal barrier β_1/x^2 together with another space-like one β_2/y^2. The coordinates (x, y) are the usual time and space ones. In contrast, expressions (4.17) and (4.20) are not well defined when $z\to 0$ since the points O_1 and O_2 go to infinity.

5. Concluding remarks

Throughout this paper we have constructed several Hamiltonians that belong to the quite general family $\mathcal{H}_z = \mathcal{H}_z(J_-, J_+, J_3)$ within the realization (1.5). There are certainly many other possible choices that lead to integrable systems on spaces of non-constant curvature with deformed $sl(2)$ coalgebra symmetry. Nevertheless, the explicit form of such a general Hamiltonian can be restricted by taking into account the following requirements: (i) the kinetic energy is quadratic in the momenta, (ii) the limit $z\to 0$ of the underlying deformed spaces leads to $\mathbf{E}^2$, (iii) dimensions of the deformation parameter are $[z] = [J_-]^{-1} = [q_i]^{-2}$, and (iv) the potential only depends on the coordinates. Under these assumptions the most general integrable Hamiltonian is just (1.8); note that the case $\mathcal{H}_z = J_3^2$ is transformed into a particular case of (1.8) through the Casimir $\mathcal{C}_z$. Hence the (deformed) kinetic energy and potential of the resulting Hamiltonian, $\mathcal{H}_z = \mathcal{T}_z + \mathcal{V}_z$, turn out to be

$$\mathcal{T}_z = \frac{1}{2}\left(\frac{\sinh zq_1^2}{zq_1^2}\,\mathrm{e}^{zq_2^2}p_1^2 + \frac{\sinh zq_2^2}{zq_2^2}\,\mathrm{e}^{-zq_1^2}p_2^2\right) f\big(z\big(q_1^2+q_2^2\big)\big) \tag{5.1}$$

$$\mathcal{V}_z = \left(\frac{zb_1}{2\sinh zq_1^2}\,\mathrm{e}^{zq_2^2} + \frac{zb_2}{2\sinh zq_2^2}\,\mathrm{e}^{-zq_1^2}\right) f\big(z\big(q_1^2+q_2^2\big)\big) + \mathcal{U}\big(z\big(q_1^2+q_2^2\big)\big). \tag{5.2}$$

We remark that the general expression for the Gaussian curvature of the 2D space associated with the kinetic energy (5.1) can be found in [12]. Then it can be checked that in order to obtain a space of constant curvature from $\mathcal{T}_z$ the choice is quite singular (e.g., $f = \mathrm{e}^{\pm zJ_-}$), and in general one obtains very involved spaces of non-constant curvature for which the simplest choice is the one we have developed in this paper with $f \equiv 1$. However other deformed spaces underlying (5.1) and particular potentials contained in (5.2) could be worth studying. Other approaches to superintegrability on 2D spaces of variable curvature can be found in [41, 42].

On the other hand, we stress that by introducing a second parameter λ_2, that determines the signature of the metric, we have been able to obtain integrable potentials on relativistic spacetimes of non-constant curvature; in this context, the deformed KC potential could be of interest in classical gravity. Furthermore the known superintegrable SW potential on Riemannian spaces has also been implemented on the three classical relativistic spacetimes of constant curvature. Note that we have avoided the contraction $\lambda_2 = 0$, which is well defined on both metrics (2.2) and (2.14), since this would give rise to degenerate (Newtonian) metrics, whose dynamical contents are not so interesting.

Finally, we recall that the existence of an underlying coalgebra symmetry for all these two-particle Hamiltonians ensures that they can be generalized to N-dimensional spaces through the coproduct. In fact, the corresponding expressions in terms of the initial phase space (q_i, p_i) can be found in [7]. Nevertheless the geometrical and physical description of the corresponding Hamiltonians on ND curved Riemannian and relativistic spaces (thus including a proper study of sectional curvatures) remains as an open problem which is currently under investigation.

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